Topology and the approximation of norms

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The norm $\|\cdot\|$ is **polyhedral** if, given any $E \subseteq X$, $\dim E < \infty$, there exist $f_1, \ldots, f_n \in S_{X^*}$ (which depend on E) such that

$$\|x\| = \max_{1 \leqslant i \leqslant n} |f_i(x)|, \qquad x \in E.$$

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4. No dual space is isomorphically polyhedral.

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$$\|x\| \leqslant |||x||| \leqslant (1+\varepsilon)\|x\| \qquad \text{for all } x \in X.$$

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- The same holds if 'polyhedral' above is replaced by 'C^k-smooth' (Hájek, Talponen 2013).
- Let Γ be a set. All norms on $c_0(\Gamma)$ can be approximated by polyhedral norms and C^∞-smooth norms (Bible, S 2016).

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In many applications, we consider sets that are both σ-w^*-LRC **and** w^*-K_σ.

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$$\{f \in B_{X^*} \;:\; |\operatorname{supp}(f)| = n\},$$

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4. If K is a σ-**discrete** compact space, then the set of Dirac measures

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   is σ-w^*-LRC and w^*-compact.
5. If E is σ-w^*-LRC and w^*-K_σ, then so is $\operatorname{span}(E)$.

The main application of such sets

Definition

Let $(X, \|\cdot\|)$ be a Banach space. A set $B \subseteq B_{X^*}$ is a (James) **boundary** of $(X, \|\cdot\|)$ if, given $x \in X$, there exists $f \in B$ such that $f(x) = \|x\|$.

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Theorem (FPST 2014, Bible 2015)

Let $(X, \|\cdot\|)$ have a boundary that is σ-w^*-LRC **and** w^*-K_σ. Then $\|\cdot\|$ can be approximated by both C^∞-smooth norms and polyhedral norms.

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Let $Y \subseteq X^*$ be an infinite-dimensional subspace. Then S_Y is not σ-w^*-LRC.

But sometimes a given norm can be approximated by norms that do have such boundaries.

A framework for approximating norms

Let Γ be an infinite set and let X support a system of non-zero projections $(P_\gamma)_{\gamma\in\Gamma}$, such that:

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2. If $X = C_0(M)$, where
$$M = \bigcup_{\gamma\in\Gamma} M_\gamma,$$
is locally compact, scattered, and the discrete union of clopen sets M_γ, set $P_\gamma f = f \cdot \mathbf{1}_{M_\gamma}$.

The function θ

Definition

Given finite $F \subseteq \Gamma$, set

$$\rho(F) = \sup\left\{ \left\| \sum_{\gamma \in F} P_\gamma^* f \right\| : f \in X^* \text{ and } \left\| P_\gamma^* f \right\| \leqslant 1 \text{ whenever } \gamma \in F \right\}.$$

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Let $p_k(f)$ be the kth largest element of $\mathrm{ran}(f) := \left\{ \left\|P_\gamma^* f\right\| \;:\; \gamma \in \Gamma \right\}$, and let

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Definition

Define $\theta : X^* \to [0, \infty]$ by

$$\theta(f) = \sum_{k=1}^{\infty} (p_k(f) - p_{k+1}(f)) \rho(G_k(f)).$$

Approximating norms on spaces having M-bases

Theorem (S, Troyanski 2018)

Let X have a shrinking bounded M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$, and let $\|\cdot\|$ have a boundary B such that $\theta(f) < \infty$ whenever $f \in B$. Then $\|\cdot\|$ can be approximated by norms having σ-w^*-LRC and w^*-K_σ boundaries.

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Corollary (S, Troyanski 2018)

Let X have a shrinking bounded M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$ and suppose $\theta(f) < \infty$ for all $f \in X^*$. Then **every** norm on X can be approximated by both C^∞-smooth norms and polyhedral norms.

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Example (Bible, S 2016)

On $c_0(\Gamma)$, $\rho(F) = |F|$ and $\theta(f) = \|f\|_1$. Hence the above applies.

1-symmetric bases

Let $(e_\gamma)_{\gamma\in\Gamma}$ be a shrinking 1-symmetric basis of X. Define

$$\lambda(n) = \left\| \sum_{k=1}^{n} e_{\gamma_k} \right\| \qquad \text{and} \qquad \mu(n) = \left\| \sum_{k=1}^{n} e^*_{\gamma_k} \right\|,$$

where $\gamma_1, \ldots, \gamma_n$ is any choice of n distinct elements of Γ.

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Proposition (S, Troyanski 2018)

Let X have a shrinking 1-symmetric basis $(e_\gamma)_{\gamma\in\Gamma}$. Then $\theta(f) < \infty$ for all $f \in X^*$ if and only if

$$\sup\left\{ \left\| \sum_{k=1}^{n} (\mu(k+1) - \mu(k)) e_{\gamma_k} \right\| \;:\; n \in \mathbb{N} \right\} < \infty,$$

where $\gamma_1, \gamma_2, \gamma_3 \ldots \in \Gamma$ are distinct (the choice is irrelevant).

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Corollary (S, Troyanski 2018)

Let X have a shrinking 1-symmetric basis $(e_\gamma)_{\gamma\in\Gamma}$. If

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or if

$$\sup\left\{\left\|\sum_{k=1}^{n}\frac{e_{\gamma_k}}{\lambda(k)}\right\| \;:\; n\in\mathbb{N}\right\} < \infty,$$

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Preduals of Lorentz spaces

Let Γ be a set and $w = (w_i) \in \ell_\infty \setminus \ell_1$ a decreasing sequence of positive numbers.

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The **predual of Lorentz space** $X := d_*(w, 1, \Gamma)$ has a shrinking 1-symmetric basis.

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We have $\theta(f) = \|f\|$ for all $f \in X^*$ and

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Hence every norm on X can be approximated by C^∞-smooth norms and polyhedral norms.

Orlicz spaces

Let Γ be a set and $M : [0, \infty) \to [0, \infty)$ a non-degenerate Orlicz function.

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if and only if M satisfies the 'summability condition':

$$\sum_{n=1}^{\infty} M\left(\frac{M^{-1}(\frac{1}{n})}{K}\right) < \infty, \qquad \text{for some } K > 1.$$

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$$\sup\left\{ \left\| \sum_{k=1}^{n} (\mu(k+1) - \mu(k)) e_{\gamma_k} \right\| \;:\; n \in \mathbb{N} \right\} < \infty,$$

if and only if M satisfies the 'summability condition':

$$\sum_{n=1}^{\infty} M\left(\frac{M^{-1}(\frac{1}{n})}{K} \right) < \infty, \qquad \text{for some } K > 1.$$

In this case every norm on X can be approximated by C^∞-smooth norms and polyhedral norms.

$C(K)$ spaces

Proposition (Marciszewski 2003)

If $C(K) \hookrightarrow c_0(\Gamma)$ for some set Γ, then $K^{(n)} = \varnothing$ for some $n \in \mathbb{N}$.

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Let α be an ordinal. There exists a compact scattered space K such that $K^{(\alpha)} \neq \varnothing$, and any norm on $C(K)$ can be approximated by C^∞-smooth norms and polyhedral norms.

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Problem

Can every norm on $C([0, \omega_1])$ be approximated by C^1-smooth norms or polyhedral norms?

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2. Can every norm on $C([0, \omega_1])$ be approximated by C^1-smooth norms or polyhedral norms?
3. Let K be a compact scattered space with $K^{(3)} = \varnothing$. Can every norm on $C(K)$ be approximated by C^2-smooth norms or polyhedral norms?